

Area spectrum in Lorentz covariant loop gravity

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Abstract

We use the manifestly Lorentz covariant canonical formalism to evaluate eigenvalues of the area operator acting on Wilson lines. To this end we modify the standard definition of the loop states to make it applicable to the present case of non-commutative connections. The area operator is diagonalized by using the usual shift ambiguity in definition of the connection. The eigenvalues are then expressed through quadratic Casimir operators. No dependence on the Immirzi parameter appears.

PACS numbers: 04.20.Fy, 04.60.-m

I. INTRODUCTION

Quantization of gravity is an extremely hard and interesting problem which remains unsolved so far. During last years a number of approaches have achieved a definite progress in treating various aspects of quantum gravity. The most elaborated and popular line of research is string theory which includes perturbative gravity in its spectrum and unifies it with other interactions. An alternative (or, perhaps, complementary) approach is the loop quantum gravity [1] (for review, see [2]). This program relies on the Dirac canonical quantization. It is explicitly nonperturbative and background independent so realizing the basic principles of general relativity. During the previous decade this approach has got rigorous mathematical foundations [3] and has led to interesting qualitative predictions about quantum spacetime.

These predictions originate from remarkable results obtained in the framework of loop quantum gravity, which are calculations of the volume and area spectra [4]- [6]. It appeared, however, that the area spectrum depends on the so called Immirzi parameter [7]. It parametrises a canonical transformation [8] which introduces a new connection field. The reason for this dependence is that this transformation cannot be realized unitarily in the

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Hilbert space of quantum theory [9]. In the language of quantum field theory this means presence of a quantum anomaly. There exist two different types of the quantum anomalies. The first type of the anomalies appear when a symmetry of the classical action cannot be preserved by quantization due to divergences or other quantum effects. Chiral and conformal anomalies belong to this type. Their presence indicates emergence of a new physics. The most celebrated example is the chiral anomaly in QCD which has been used for description of the low energy hadron physics since late 60's. Rather naturally, it has been suggested [9] that the anomaly in the mentioned canonical transformation belongs to this type and, consequently, the Immirzi parameter is a new fundamental constant.

One cannot however exclude the second possibility. An anomaly could appear if a symmetry is involuntary broken by the choice of a particular quantization scheme. If this is the case, the remedy can be in applying another quantization scheme which explicitly preserves as much important symmetries as possible. This is the route we take in the present paper by applying the manifestly Lorentz covariant quantization of [10] to calculation of the area spectrum.

There are already some evidences that the Immirzi parameter dependence may disappear in a more symmetric quantization scheme. In the paper [10] the path integral quantization scheme of [11] has been extended to arbitrary values of the Immirzi parameter. It has been demonstrated, that the Immirzi parameter dependence does not appear in the path integral. We should stress, that in principle the path integral formalism is capable to see non-perturbative effects (as e.g. the virtual black hole formation [12]). Another important result was obtained recently by Samuel [13] who demonstrated that the Barbero connection is not a Lorentz connection.

Recently, the importance for the theory to be Lorentz-covariant has been also recognized in spin foam models [14] which represent the modern development of loop quantum gravity [15]. However, the Lorentz covariance has been introduced there without any reference to the canonical quantization. It is an important task to develop a Lorentz covariant formulation "from the first principles".

In this paper we apply the Lorentz covariant canonical quantization developed in [10] to loop quantum gravity. We re-derive the spectrum of the area operator in the new framework. To this end we construct the Wilson line operator with true Lorentz connection. Since the Dirac brackets of the connections are non-zero, there is not connection representation. However, by choosing an appropriate vacuum state we are able to construct the quantum states corresponding to the Wilson lines which behave in a very similar way to the ordinary loop states. However, the area operator is not necessarily diagonal on these states. To diagonalize this operator we use the usual ambiguity in the connection: any connection can be shifted by a vector and will still remain a proper connection. It appears, that the shift is uniquely defined by the requirements that it vanishes on the constraint surface and that the area operator is diagonal on the Wilson line states. This new connection obeys a remarkably simple bracket algebra. Eigenvalues of the area operator are then calculated. They *do not* depend on the Immirzi parameter.

The paper is organized as follows. In the next section we summarize the covariant canonical formulation of [10]. In sec. III we discuss the choice of the connection variables to be used in the Wilson line states. Area spectrum is calculated in sec. IV. Section V is devoted to discussion of the results, problems and future perspectives. Appendices are intended to list various definitions and useful properties.

We use the following notations for indices. The indices i, j, \dots from the middle of the

alphabet label the space coordinates. The latin indices a, b, \dots from the beginning of the alphabet are the $so(3)$ indices, whereas the capital letters X, Y, \dots from the end of the alphabet are the $so(3, 1)$ indices.

II. $SO(3, 1)$ -COVARIANT CANONICAL FORMULATION

In this section we review the covariant formalism developed in [10]. It is a canonical formulation of general relativity based on the generalized Hilbert–Palatini action suggested by Holst [16]

$$S_{(\beta)} = \frac{1}{2} \int \varepsilon_{\alpha\beta\gamma\delta} e^\alpha \wedge e^\beta \wedge (\Omega^{\gamma\delta} + \frac{1}{\beta} \star \Omega^{\gamma\delta}). \quad (1)$$

Here the star operator is defined as $\star \omega^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta}{}_{\gamma\delta} \omega^{\gamma\delta}$, and $\Omega^{\alpha\beta}$ is the curvature of the spin-connection $\omega^{\alpha\beta}$. A $3 + 1$ decomposition of the fields reads:

$$\begin{aligned} e^0 &= N dt + \chi_a E_a^i dx^i, & e^a &= E_a^i dx^i + E_i^a N^i dt, \\ \tilde{E}_a^i &= h^{1/2} E_a^i, & \tilde{N} &= h^{-1/2} N, & \sqrt{h} &= \det E_i^a, \\ N^i &= \mathcal{N}_D^i + \tilde{E}_a^i \chi^a \mathcal{N}, & \tilde{N} &= \mathcal{N} + \tilde{E}_i^a \chi_a \mathcal{N}_D^i. \end{aligned} \quad (2)$$

Here E_a^i is the inverse of E_i^a . The field χ_a describes deviation of the normal to the spacelike hypersurface $\{t = 0\}$ from the time direction.

Let us introduce matrix fields carrying one Lorentz index

$$\begin{aligned} A^X &= (\tfrac{1}{2} \omega^{0a}, \tfrac{1}{2} \varepsilon_a{}^{bc} \omega^{bc}) - \text{connection multiplet}, \\ \tilde{P}_X^i &= (\tilde{E}_a^i, \varepsilon_a{}^{bc} \tilde{E}_b^i \chi_c) - \text{first triad multiplet}, \\ \tilde{Q}_X^i &= (-\varepsilon_a{}^{bc} \tilde{E}_b^i \chi_c, \tilde{E}_a^i) - \text{second triad multiplet}, \\ \tilde{P}_{(\beta)X}^i &= \tilde{P}_X^i - \frac{1}{\beta} \tilde{Q}_X^i - \text{canonical triad multiplet}, \end{aligned} \quad (3)$$

which form multiplets in the adjoint representation of $so(3, 1)$. In Appendix A we present the relations between the triad multiplets and introduce the numerical matrices Π and R (A2), (A3) appearing in the formulas below. In terms of these fields the decomposed action can be represented in the form:

$$\begin{aligned} S_{(\beta)} &= \int dt d^3x (\tilde{P}_{(\beta)X}^i \partial A_i^X + \mathcal{N}_G^X \mathcal{G}_X + \mathcal{N}_D^i H_i + \mathcal{N} H), \\ \mathcal{G}_X &= \partial_i \tilde{P}_{(\beta)X}^i + f_{XY}^Z A_i^Y \tilde{P}_{(\beta)Z}^i, \\ H_i &= -\tilde{P}_{(\beta)X}^j F_{ij}^X, \\ H &= -\frac{1}{2(1 + \frac{1}{\beta^2})} \tilde{P}_{(\beta)X}^i \tilde{P}_{(\beta)Y}^j f_Z^{XY} R_W^Z F_{ij}^W, \\ F_{ij}^X &= \partial_i A_j^X - \partial_j A_i^X + f_{YZ}^X A_i^Y A_j^Z, \end{aligned} \quad (4)$$

where f_{XY}^Z are $so(3, 1)$ structure constants, $\mathcal{N}_G^X = A_0^X$. The $so(3, 1)$ indices are raised and lowered with the help of the Killing form

$$g_{XY} = \frac{1}{4} f_{XZ_1}^{Z_2} f_{YZ_2}^{Z_1}, \quad g^{XY} = (g^{-1})^{XY}, \quad g_{XY} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & -\delta_{ab} \end{pmatrix}. \quad (5)$$

The limit $\beta \rightarrow i$ gives Ashtekar gravity. Even though the Hamiltonian constraint H in (4) has apparently a pole at $\beta = i$ one can demonstrate [10] that this limit is non-singular.

The canonical variables of the model are A_i^X and $\tilde{P}_{(\beta)X}^i$. G_X , H_i and H are first class constraints obeying the algebra presented in Appendix C. We call them the Gauss law, diffeomorphism and Hamiltonian constraints respectively. There are also two sets of the second class constraints. They are represented by 3×3 symmetric fields

$$\phi^{ij} = \Pi^{XY} \tilde{Q}_X^i \tilde{Q}_Y^j = 0, \quad (6)$$

$$\psi^{ij} = f^{XYZ} \tilde{Q}_X^{[l} \tilde{Q}_Y^{\{j\}} \partial_l \tilde{Q}_Z^{i\}} - 2(\tilde{Q}\tilde{Q})^{\{i[j} \tilde{Q}_Z^{l\}} A_l^Z = 0, \quad (7)$$

$$(\tilde{Q}\tilde{Q})^{ij} = g^{XY} \tilde{Q}_X^i \tilde{Q}_Y^j. \quad (8)$$

Symmetrization is taken with the weight 1/2. Antisymmetrization includes no weight.

The existence of the second class constraints gives rise to the Dirac bracket [17]

$$\{K, L\}_D = \{K, L\} - \{K, \varphi_r\}(\Delta^{-1})^{rr'} \{\varphi_{r'}, L\}, \quad (9)$$

where $\varphi_r = (\phi^{ij}, \psi^{ij})$. The matrix of commutators of the second class constraints $\Delta^{rr'}$ can be found in Appendix B. Both Δ and Δ^{-1} are triangular. Due to this when one of the functions in (9), K or L is a first class constraint, the Dirac bracket coincides with the ordinary one (except for the case when $K = H$ and L depends on the connection). In particular, this gives

$$\begin{aligned} \{\mathcal{G}_X, \mathcal{G}_Y\}_D &= f_{XY}^Z \mathcal{G}_Z, \\ \{\mathcal{G}_X, A_i^Y\}_D &= \delta_X^Y \partial_i - f_{XZ}^Y A_i^Z, \\ \{\mathcal{G}_X, \tilde{P}_Y^i\}_D &= f_{XY}^Z \tilde{P}_Z^i. \end{aligned} \quad (10)$$

Finally, the Dirac brackets of the canonical variables have the form:

$$\begin{aligned} \{\tilde{P}_{(\beta)X}^i, \tilde{P}_{(\beta)Y}^j\}_D &= 0, \\ \{A_i^X, \tilde{P}_{(\beta)Y}^j\}_D &= \delta_i^j \delta_Y^X - \frac{1}{2} R^{XZ} \left(\tilde{Q}_Z^j Q_i^W + \delta_i^j I_{(Q)Z}^W \right) g_{WY}, \\ \{A_i^X, A_j^Y\}_D &= -\{A_i^X, \phi^{kl}\} (D_1^{-1})_{(kl)(mn)} \{\psi^{mn}, A_r^Z\} \{\tilde{P}_{(\beta)Z}^r, A_j^Y\}_D \\ &\quad - \{A_i^X, \tilde{P}_{(\beta)Z}^r\}_D \{A_r^Z, \psi^{mn}\} (D_1^{-1})_{(mn)(kl)} \{\phi^{kl}, A_j^Y\}. \end{aligned} \quad (11)$$

Here Q_i^X is the inverse triad multiplet and

$$I_{(P)X}^Y := \tilde{P}_X^i P_i^Y, \quad I_{(Q)X}^Y := \tilde{Q}_X^i Q_i^Y \quad (12)$$

are projectors on \tilde{Q} and \tilde{P} -multiplets (see Appendix B for details).

Quantization may go along the usual way. We may replace the canonical variables by operators and define a commutator on them as $[\cdot, \cdot] := i\hbar \{\cdot, \cdot\}_D$. Of course, when we replace the canonical variables by operators, the right hand side of (11) becomes ambiguous. In actual calculations of the area spectrum we will use a shifted connection \mathcal{A} . As we will see in section III B, for this connection no ordering ambiguity appears.

III. AREA OPERATOR AND THE WILSON LINE

A. Wilson line with canonical connection

In [10] it was suggested to use the Lorentz covariant formulation described above as a basis for a modified loop approach. The key point is that A_i^X is a true Lorentz connection (10) and so one can construct the Wilson line operator

$$\hat{U}_\alpha(a, b) = \mathcal{P} \exp \left(\int_a^b dx^i A_i^X T_X \right), \quad (13)$$

where α is a path between two points a and b , T_X is a gauge generator. However, we encounter a serious obstacle since instead of simple standard canonical commutation relations now we have a complicated algebra of the Dirac brackets (11). In particular, the operators like (13) fail to form the loop algebra. Moreover, since the connection A_i^X is non-commutative the connection representation does not exist.

Nevertheless, one might hope to obtain some results relying on the bracket algebra (11) only. Let us try to obtain the spectrum of the area operator extensively investigated in the framework of the standard loop approach [4,5]. Here we follow the line of reasonings suggested in [2]. In particular, we use the same regularization technique for the area operator. Namely, define the operator of the triad smeared over a two-dimensional surface embedded in the 3-manifold:

$$\tilde{P}_X(\Sigma) = \int_\Sigma d^2\sigma n_i(\sigma) \tilde{P}_X^i(\sigma), \quad (14)$$

where the embedding is described by the coordinates $x^i(\vec{\sigma})$ and the normal to the surface is given by $n_i = \varepsilon_{ijk} \frac{\partial x^j}{\partial \sigma^1} \frac{\partial x^k}{\partial \sigma^2}$. Then the regularized area operator is defined as follows:

$$\mathcal{S} = \lim_{\rho \rightarrow \infty} \sum_n \sqrt{g(S_n)}, \quad (15)$$

where the sum is taken over a partition ρ of S into small surfaces S_n , $\bigcup_n S_n = S$, and ¹

$$g(\Sigma) = g^{XY} \tilde{P}_X(\Sigma) \tilde{P}_Y(\Sigma). \quad (16)$$

We define a state vector corresponding to the Wilson line operator \hat{U}_α as

$$U_\alpha = \hat{U}_\alpha |0\rangle, \quad (17)$$

where $|0\rangle$ is a vacuum state. To be as close as possible to the connection representation formalism, we require

$$\tilde{P}_X^i |0\rangle = 0. \quad (18)$$

¹Being expressed through $\tilde{P}_{(\beta)}$ the operator $g(\Sigma)$ reads: $\beta^2 g^{XY} \tilde{P}_{(\beta)X}(\Sigma) \tilde{P}_{(\beta)Y}(\Sigma) / (\beta^2 - 1)$ The printed version of [10] contains a mistake in this formula.

Since \tilde{P}_X^i are commutative, the condition (18) is consistent. The condition (18) may lead to troubles if one acts by the inverse triad on the vacuum state. To avoid problems one may consider a more general vacuum state with a non-trivial internal geometry

$$\tilde{P}_X^i|0\rangle = \langle\tilde{P}_X^i\rangle|0\rangle. \quad (19)$$

Consistency with the second class constraints requires that $\langle\tilde{P}_X^i\rangle$ is expressed through $\langle\tilde{E}\rangle$ and $\langle\chi\rangle$ as in (3). After the calculations one can take $\langle\tilde{P}_X^i\rangle \rightarrow 0$. The vacuum state (19) may be also interesting on its own right (see discussion in section V). We shall use primarily the simplest vacuum (18), but shall also comment at some points which modifications would appear if the vacuum (19) is used instead.

We have constructed a natural generalization of the the Wilson line states for the case of non-commutative Lorentz connection. Let us recall that the unitary representations of the Lorentz group are infinite dimensional. Therefore, it is much harder to address orthogonality, completeness and other functional properties of the loop states than in the standard $su(2)$ case. We will not discuss these properties here. Instead, we concentrate on the algebraic aspect of the problem.

To find the area spectrum we study the action of the smeared triad on a state created by the Wilson line. Consider the simplest situation when the path α has with the surface Σ one intersecting point c which breaks α in two parts, α_1 and α_2 . Then the action is given by

$$\begin{aligned} \tilde{P}_X(\Sigma)\hat{U}_\alpha(a,b)|0\rangle = & - \int_\Sigma d^2\sigma \int_\alpha ds \varepsilon_{ijl} \frac{\partial x^i}{\partial \sigma^1} \frac{\partial x^j}{\partial \sigma^2} \frac{\partial x^k}{\partial s} \delta^3(\vec{x}(\sigma), \vec{x}(s)) \\ & \times \hat{U}_{\alpha_1}(a,c)[A_k^Y T_Y, \tilde{P}_X^l]\hat{U}_{\alpha_2}(c,b)|0\rangle. \end{aligned} \quad (20)$$

Here the vacuum state (18) has been used. For the vacuum (19) an additional term $\langle\tilde{P}_{(\beta)X}(\Sigma)\rangle\hat{U}_\alpha(a,b)$ appears on the right hand side of (20).

In the standard loop approach [4,5] one has to consider the action of the smeared triad \tilde{E} on the Wilson line with $su(2)$ connection A_i^a . Therefore, the equation (20) should be replaced by an analogous one with the commutator of the canonical variables $[A_i^a, \tilde{E}_b^j]$ on the right hand side. This commutator is proportional to δ_i^j . Because of this fact, the explicit x -dependence can be canceled, and the right hand side of $\tilde{E}U_\alpha$ becomes in the standard loop approach a purely algebraic expression. As a result the area operator (that is essentially \tilde{E} applied twice) can be easily diagonalized. In the present case $\{A_k^Y, \tilde{P}_X^l\}_D$ is *not* proportional to δ_k^l . Consequently, the area operator acting on the Wilson line U_α with the canonical connection A is not just a matrix in the Lorentz indices and cannot be that easily made diagonal. A way to by-pass this difficulty is suggested in the next section III B.

B. Shifted connection

We have seen that to enable diagonalization the area operator the commutator of the connection and P should be unit matrix in the spatial indices. It is known that if one adds a vector to a connection the resulting object will again transform as a connection. We are going to use this arbitrariness in the choice of the connection to diagonalize the area operator. We are interested in a new connection \mathcal{A}_i^X such that: i) it is a true Lorentz connection, i.e. $\mathcal{A}_i^X - A_i^X$ is tensorial in both indices; ii) the Dirac bracket $\{\mathcal{A}_k^Y, \tilde{P}_X^l\}_D$ is proportional to δ_k^l ; iii) $\mathcal{A}_i^X - A_i^X$ is proportional to the first class constraints. These requirements appear to

be very strong. There is just one connection which satisfies all of them. To show this, let us note that all the triad (or tetrad) components have dimension zero, while the connection has mass dimension one. Consequently the Gauss constraint has dimension one, and the diffeomorphism and Hamiltonian constraints have dimension two. It is clear therefore that

$$\mathcal{A}_i^X = A_i^X + \alpha_i^{XY}(Q)\mathcal{G}_Y, \quad (21)$$

where $\alpha_i^{XY}(Q)$ does not contain derivatives or connections. The coefficient functions $\alpha_i^{XY}(Q)$ have to be tensorial in order to ensure correct diffeomorphism and Lorentz transformation properties of \mathcal{A} :

$$\{\mathcal{G}_X, \mathcal{A}_i^Y\}_D = \delta_X^Y \partial_i - f_{XZ}^Y \mathcal{A}_i^Z, \quad (22)$$

$$\{\mathcal{D}(\vec{N}), \mathcal{A}_i^X\}_D = \mathcal{A}_j^X \partial_i N^j + N^j \partial_j \mathcal{A}_i^X. \quad (23)$$

$\mathcal{D}(\vec{N})$ is defined in Appendix C (C1). Thus \mathcal{A}_i^X is the true $\text{so}(3,1)$ connection. There is still a 6-parameter family of the connections which satisfy (22) and (23). This ambiguity is fixed uniquely by the second condition ii). We arrive at the following Lorentz connection:

$$\mathcal{A}_i^X = A_i^X + \frac{1}{2\left(1 + \frac{1}{\beta^2}\right)} R_S^X I_{(Q)}^{ST} R_T^Z f_{ZW}^Y \mathcal{P}_i^W \mathcal{G}_Y. \quad (24)$$

The connection \mathcal{A}_i^X has a very simple bracket with \tilde{P}_Y^j

$$\{\mathcal{A}_i^X, \tilde{P}_Y^j\}_D = \delta_i^j I_{(P)Y}^X. \quad (25)$$

Already at this point we observe independence of the right hand side of (25) from β . It should be stressed that this β -independence is *not* a pre-requirement in our construction. This is rather a consequence of the conditions i)–iii) above. We observe also

$$\{\mathcal{A}_i^X, \mathcal{P}_j^Y\}_D = -\mathcal{P}_j^X \mathcal{P}_i^Y, \quad (26)$$

$$\{\mathcal{A}_i^X, I_{(P)}^{YZ}\}_D = 0. \quad (27)$$

Due to this relation the projectors $I_{(P)}$ and $I_{(Q)}$ behave very similar to c -numbers.

The Dirac bracket of two connections has a very complicated form and will not be presented here. However, an important observation can be made already by considering the Jacobi identity

$$\{\{\mathcal{A}_i^X, \mathcal{A}_j^Y\}_D, \tilde{P}_Z^k\}_D = \{\{\mathcal{A}_i^X, \tilde{P}_Z^k\}_D, \mathcal{A}_j^Y\}_D - \{\{\mathcal{A}_j^Y, \tilde{P}_Z^k\}_D, \mathcal{A}_i^X\}_D = 0. \quad (28)$$

It follows from (28) that $\{\mathcal{A}_i^X, \mathcal{A}_j^Y\}_D$ does not depend on the connection. It is a function of \tilde{Q} and its derivatives, i.e. this bracket contains only commuting objects on the right hand side. Therefore, there will be no ordering ambiguity if we replace the Dirac brackets with \mathcal{A}_i^X by the corresponding operator relation. We will use this as a new quantization rule. In particular,

$$[\mathcal{A}_i^X, \tilde{P}_Y^j] = i\hbar \delta_i^j I_{(P)Y}^X. \quad (29)$$

Note, that the commutators with the new connection (24) are insufficient to define all commutators involving the canonical connection. The reason is that the (classical) field \mathcal{A}_i^X satisfies the condition

$$g_{YZ}(\delta_i^k I_{(Q)X}^Y - Q_i^Y \tilde{Q}_X^k) \mathcal{A}_k^Z = I_{(Q)X}^Y f_{YZ}^W Q_i^Z \partial_j \tilde{Q}_W^j \quad (30)$$

and has fewer independent components than A_i^X . From (24) it is clear that the missing components are contained in the Gauss constraint. For practical purposes it is therefore enough to know the commutators with \mathcal{A}_i^X and the commutators with the Gauss constraint which are defined either by the structure constants of the Lorentz group or by the matrix elements in corresponding representations. These quantization rules have one more important advantage. They ensure that quantum transformation laws are identical to the classical ones. So there will be no gauge anomaly for the Lorentz group.

IV. AREA SPECTRUM

The shifted connection \mathcal{A} can be used as an argument the Wilson line. Let us evaluate action of the area operator (15) on the states created by such Wilson lines. It is given by

$$\mathcal{S}\hat{U}_\alpha[\mathcal{A}]|0\rangle = \hbar\hat{U}_{\alpha_1}[\mathcal{A}]\sqrt{-I_{(P)}^{XY}T_X T_Y}\hat{U}_{\alpha_2}[\mathcal{A}]|0\rangle, \quad (31)$$

where we used equations (20) and (29) and the prescription [4,5] for taking the square root of the operator (assuming that the latter is still valid for the Lorentz gauge group). Vacuum state is supposed to be the trivial one (18).

Consider the matrix operator $I_{(P)}^{XY}T_X T_Y$. It can be rewritten as

$$I_{(P)}^{XY}T_X T_Y = g^{XY}T_X T_Y - I_{(Q)}^{XY}T_X T_Y, \quad (32)$$

where the first term is a quadratic Casimir of the Lorentz algebra:

$$g^{XY}T_X T_Y = C_2(so(3,1)). \quad (33)$$

In order to study the second term in (32) let us introduce the generators

$$q_a := \frac{1}{\sqrt{1-\chi^2}} \left(\delta_{ab} - \frac{1-\sqrt{1-\chi^2}}{\chi^2} \chi_a \chi_b \right) E_i^b \tilde{Q}_X^i T^X. \quad (34)$$

One can check directly that

$$I_{(Q)}^{XY}T_X T_Y = -q_a q_a, \quad (35)$$

$$[q_a, q_b] = -\varepsilon_{ab}^c q_c. \quad (36)$$

Consequently, q_a generate the $so(3)$ subalgebra of $so(3,1)$, and $I_{(Q)}^{XY}T_X T_Y$ is the Casimir operator of this subalgebra

$$q_a q_a = -C_2(so(3)). \quad (37)$$

In a suitable basis in the defining representation of $so(3,1)$ the generators q_a annihilate the vector $v_\chi = (1-\chi^2)^{-1/2}(1, \chi_a)$. All vectors v_χ belong to the same orbit of the Lorentz group. Therefore, the subalgebras spanned by $\{q_a\}$ for different χ are conjugate in $so(3,1)$, and spectrum of $so(3)$ representations obtained after the restriction $so(3,1) \downarrow so(3)$ from a given representation of $so(3,1)$ does not depend on χ . Eigenvalues of the Casimir operator (37) are also χ -independent.

Spectrum of the area operator acting on Wilson lines reads:

$$\mathcal{S} \sim \hbar \sqrt{-C_2(\mathfrak{so}(3,1)) + C_2(\mathfrak{so}(3))}. \quad (38)$$

This formula represents the main result of our paper.

One can think naively that the Lorentz invariance of the area spectrum (38) is broken due to the presence of the Casimir operator of a subgroup. This is however not the case. Under local Lorentz transformations the Wilson line changes as $U(x, y) \rightarrow \mathcal{U}(x)U(x, y)\mathcal{U}^{-1}(y)$, where $\mathcal{U}(x)$ is an element of the Lorentz group taken in an appropriate representation. The matrix operator $\sqrt{-I_{(P)}^{XY}T_X T_Y}$ changes in a similar way: $\sqrt{-I_{(P)}^{XY}T_X T_Y} \rightarrow \mathcal{U}(x)\sqrt{-I_{(P)}^{XY}T_X T_Y}\mathcal{U}^{-1}(x)$. Thus proper (covariant) transformation properties of (31) are recovered.

As expected, the area spectrum (38) does not depend on the Immirzi parameter β .

V. DISCUSSION

In this paper we analysed the area operator spectrum in a manifestly Lorentz covariant formalism. We have constructed a generalization of the Wilson line states for the case of non-commutative connection. As usual, there is certain arbitrariness in the choice of the connection. Namely, any connection can be shifted by a vector and would still remain a connection. We use this arbitrariness to define a connection \mathcal{A} such that $\{\mathcal{A}_i^X, \tilde{P}_Y^j\}_D \sim \delta_i^j$. Because of the rather simple commutation relations (29) we are able to find explicitly the area spectrum (38). Since the right hand side of (29) does not depend on the Immirzi parameter β , there is no dependence on β in the spectrum (38) as well.

Note, that the connection \mathcal{A} is unique only if we require that it coincides with A on the surface of the constraints. A different idea might be to fix the connection by considering its space-time properties. Because of the rather complicated form of the Dirac brackets with the Hamiltonian constraint this is a technically very involved calculation. We may hope that the results obtained in this way will agree with our results.

We must admit that there is no proof in this paper that the area spectrum with *any* connection does not depend on β . We cannot perform direct calculations with a connection other than \mathcal{A} . We may, however, *interpret* the shift $A \rightarrow \mathcal{A}$ as diagonalization of the area operator. Our results suggest that in a Lorentz covariant quantization the dependence of the physical quantities on the Immirzi parameter ultimately disappears.

In addition to the explicit Lorentz covariance there is another advantage of our approach. The Hamiltonian constraint (4) is polynomial in the canonical variables (as for the Ashtekar or Euclidean cases). Due to this the corresponding regularized quantum operator may be similar to the first term of Thiemann's constraint operator [18]. That would eliminate difficulties created by the second term. Note that the spin foam formulation of loop quantum gravity takes into account the first term of Thiemann's Hamiltonian only [2,15].

Let us comment on the choice of the vacuum state. The connection representation implies that the trivial vacuum (18) is chosen. Such representation does not exist in our case due to the non-commutativity of the connection fields. Therefore, we must choose a vacuum state explicitly. The possibility of a more general vacuum state (19) can be taken into account. (A similar possibility has been already discussed in [19]). For the vacuum state (19) we have no problem with the action of the inverse triad on the vacuum, but we loose explicit background independence. Physical consequences of different vacua have to be clarified yet.

Even without relation to the Immirzi parameter problem quantization of gravity in manifestly Lorentz invariant terms is an important task. We have considered here the algebraic part of the problem, while the functional analysis part has been completely ignored. We do not know how to construct a complete orthogonal basis in the space of states out of the Wilson lines. Consequently, we may only guess which representations do actually contribute to the area spectrum (38).

The area spectrum (38) now contains the Casimir operator of non-compact Lorentz group. Since unitary representations of the Lorentz group are labelled by a pair of indices (ρ, j) , and the index ρ is continuous, we may expect that the area spectrum becomes continuous as well. This would be a new feature for the loop quantum gravity, though continuous spectrum appears in the spin foam models [14]. However, in the view of the remarks in the previous paragraph, this feature should be taken with great amount of care.

Recently, a manifestly $so(3,1)$ -covariant formalism has been developed in the framework of spin foam models [14]. It has been suggested to use the so-called simple representations of the Lorentz group only. The Immirzi parameter has been also included in this approach [20,21]. The area spectrum obtained in the spin foam models is different from our expression (38). The reason is that we use different quantization rules. We should stress that our commutation relations are *derived* from the gravitational action rather than postulated. Therefore, our quantization rules may provide a more solid ground for the Lorentz-invariant spin foam models. Despite of complicated Dirac brackets our final commutation relations (29) are rather simple. It should be possible to use them in the spin foam approach.

ACKNOWLEDGEMENTS

We are grateful to Abhay Ashtekar for fruitful discussions. This work of D.V. has been partially supported by the DFG project Bo 1112/11-1. The research of S.A. has been supported in part by European network EUROGRID HPRN-CT-1999-00161.

APPENDIX A: MATRIX ALGEBRA

In the basis (3) the $so(3,1)$ structure constants are:

$$\begin{aligned} f_{A_1 A_2}^{A_3} &= 0, & f_{A_1 B_2}^{A_3} &= -\varepsilon^{A_1 B_2 A_3}, & f_{B_1 B_2}^{A_3} &= 0, \\ f_{B_1 B_2}^{B_3} &= -\varepsilon^{B_1 B_2 B_3}, & f_{A_1 B_2}^{B_3} &= 0, & f_{A_1 A_2}^{B_3} &= \varepsilon^{A_1 A_2 B_3}. \end{aligned} \quad (A1)$$

Here we split the 6-dimensional index X into a pair of 3-dimensional indices, $X = (A, B)$, so that $A, B = 1, 2, 3$. ε is the Levi-Civita symbol, $\varepsilon^{123} = 1$.

All triad multiplets are connected by numerical matrices:

$$\tilde{P}_X^i = \Pi_X^Y \tilde{Q}_Y^i, \quad \Pi_X^Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta_a^b, \quad (A2)$$

$$\tilde{P}_X^i = \frac{R_X^Y}{1 + \frac{1}{\beta^2}} \tilde{P}_{(\beta)Y}^i, \quad R_X^Y = \begin{pmatrix} 1 & -\frac{1}{\beta} \\ \frac{1}{\beta} & 1 \end{pmatrix} \delta_a^b. \quad (A3)$$

They as well as their inverse commute with each other and, furthermore, they commute with the structure constants in the following sense:

$$f^{XYZ'}\Pi_{Z'}^Z = f^{XY'Z}\Pi_{Y'}^Y. \quad (\text{A4})$$

Other useful relations can be found in [10].

APPENDIX B: INVERSE MULTIPLETS AND PROJECTORS

The inverse triad multiplets are introduced as the following fields:

$$\begin{aligned} \underline{P}_i^X &= \left(\frac{\delta_b^a - \chi^a \chi_b}{1 - \chi^2} \underline{E}_i^b, -\frac{\varepsilon_{bc}^a \underline{E}_i^b \chi^c}{1 - \chi^2} \right), \\ \underline{Q}_i^X &= \left(\frac{\varepsilon_{bc}^a \underline{E}_i^b \chi^c}{1 - \chi^2}, \frac{\delta_b^a - \chi^a \chi_b}{1 - \chi^2} \underline{E}_i^b \right). \end{aligned} \quad (\text{B1})$$

They satisfy:

$$\{\mathcal{G}_X, \underline{P}_i^Y\} = -f_{XZ}^Y \underline{P}_i^Z, \quad \tilde{P}_X^i \underline{P}_j^X = \delta_j^i, \quad \tilde{Q}_X^i \underline{P}_j^X = 0. \quad (\text{B2})$$

Similar properties are valid for $\underline{Q}_i^X = \Pi_Y^X \underline{P}_i^Y$.

The projectors (12) read:

$$I_{(P)X}^Y = \begin{pmatrix} \frac{\delta_a^b - \chi_a \chi^b}{1 - \chi^2} & \frac{\varepsilon_a^{bc} \chi_c}{1 - \chi^2} \\ \frac{\varepsilon_a^{bc} \chi_c}{1 - \chi^2} & -\frac{\delta_a^b \chi^2 - \chi_a \chi^b}{1 - \chi^2} \end{pmatrix} \quad (\text{B3})$$

and $I_{(Q)X}^Y = \delta_X^Y - I_{(P)X}^Y$. Besides, one can note the relations which are very helpful in calculations:

$$I_{(P)}^{XY} = -\Pi_Z^X I_{(Q)}^{ZW} \Pi_W^Y, \quad (\text{B4})$$

$$f^{WYZ} I_{(P)W}^X \tilde{Q}_Y^i \tilde{Q}_Z^j = 0, \quad (\text{B5})$$

$$f^{WYZ} I_{(Q)W}^X \tilde{Q}_Y^i \tilde{Q}_Z^j = f^{XYZ} \tilde{Q}_Y^i \tilde{Q}_Z^j. \quad (\text{B6})$$

The commutators of the second class constraints form the following triangular matrix:

$$\Delta = \begin{pmatrix} 0 & D_1 \\ -D_1 & D_2 \end{pmatrix}, \quad \Delta^{-1} = \begin{pmatrix} D_1^{-1} D_2 D_1^{-1} & -D_1^{-1} \\ D_1^{-1} & 0 \end{pmatrix}, \quad (\text{B7})$$

where

$$D_1^{(ij)(kl)} = \{\phi^{ij}, \psi^{kl}\} = \frac{4\beta^2}{1 + \beta^2} (\tilde{Q}\tilde{Q})^{\{i[j}\} (\tilde{Q}\tilde{Q})^{\{k]l\}}, \quad (\text{B8})$$

$$(D_1^{-1})_{(kl)(mn)} = \frac{1}{8} \left(1 + \frac{1}{\beta^2} \right) ((QQ)_{kl} (QQ)_{mn} - (QQ)_{km} (QQ)_{ln} - (QQ)_{kn} (QQ)_{lm}). \quad (\text{B9})$$

Explicit form of D_2 is not needed since all brackets are expressed in terms of D_1^{-1} only (see (11)).

APPENDIX C: CONSTRAINT ALGEBRA

Define the smeared constraints:

$$\begin{aligned}\mathcal{G}(n) &= \int d^3x n^X \mathcal{G}_X, & H(\underline{N}) &= \int d^3x \underline{N} H, \\ \mathcal{D}(\vec{N}) &= \int d^3x N^i (H_i + A_i^X \mathcal{G}_X).\end{aligned}\tag{C1}$$

They obey the following algebra:

$$\begin{aligned}\{\mathcal{G}(n), \mathcal{G}(m)\}_D &= \mathcal{G}(n \times m), \\ \{\mathcal{D}(\vec{N}), \mathcal{D}(\vec{M})\}_D &= -\mathcal{D}([\vec{N}, \vec{M}]), \\ \{\mathcal{D}(\vec{N}), \mathcal{G}(n)\}_D &= -\mathcal{G}(N^i \partial_i n), \\ \{H(\underline{N}), \mathcal{G}(n)\}_D &= 0, \\ \{\mathcal{D}(\vec{N}), H(\underline{N})\}_D &= -H(\mathcal{L}_{\vec{N}} \underline{N}), \\ \{H(\underline{N}), H(\underline{M})\}_D &= \mathcal{D}(\vec{K}) - \mathcal{G}(K^j A_j),\end{aligned}\tag{C2}$$

where

$$\begin{aligned}(n \times m)^X &= f_Y^X n^Y m^Z, & \mathcal{L}_{\vec{N}} \underline{N} &= N^i \partial_i \underline{N} - \underline{N} \partial_i N^i, \\ [\vec{N}, \vec{M}]^i &= N^k \partial_k M^i - M^k \partial_k N^i, \\ K^j &= (\underline{N} \partial_i \underline{M} - \underline{M} \partial_i \underline{N}) \tilde{Q}_X^i \tilde{Q}_Y^j g^{XY}.\end{aligned}\tag{C3}$$

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